*J. Fluid Mech.* (1990), *vol.* 211, *pp.* 617–627
*Printed in Great Britain*


# On periodic and solitary wavelike solutions of the intermediate long-wave equation

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(Received 21 February 1989 and in revised form 28 July 1989)

The intermediate long-wave (ILW) equation is a weakly nonlinear integro-differential equation which governs the evolution of long internal waves in a stratified fluid of finite depth. It reduces to the Korteweg–de Vries (KdV) and to the Benjamin–Ono (BO) equations for shallow and large depths respectively. Solitary wave solutions of the ILW equation are well known, however analytic expressions for periodic solutions of the same equation do not seem to exist. Such expressions are derived in this paper and a remarkable property discovered for these periodic waves is that they can be represented as an infinite sum of spatially repeated solitons. Thus, *nonlinear* periodic solutions of the ILW equation are obtained by *linear* superposition of solitons.

## 1. Introduction

Recently there has been considerable interest in the so-called intermediate long-wave equation as a model equation for wave propagation in localized regions of waveguides, such as ocean pynoclines, atmospheric fronts or regions of temperature inversion (Lipovskiy 1986). The intermediate long-wave equation (ILW) (1) is an important singular integro-differential equation which arises in the context of long internal gravity waves in a stratified fluid with finite depth (Kubota, Ko & Dobbs 1978; Koop & Butler 1981; Segur & Hammack 1982; Santini, Ablowitz & Fokas 1984). It appears in other circumstances as well, e.g. large-scale solitary wave motion in both the atmosphere (Christie, Muirhead & Hales 1978) and the ocean (Osborne & Burch 1980; Liu, Holbrook & Apel 1985), long waves in a stratified shear flow (Maslowe & Redekopp 1980) or in atmospheric layers of drastic wind velocity changes (Romanova 1984). The ILW equation was also found to arise naturally in the recent study of the classical, yet not wholly understood, *dead water phenomenon* (Miloh & Tulin 1988) which was first discussed by Ekman (1904).

The ILW equation is a weakly nonlinear evolution equation which displays a balance between nonlinearity and dispersion. For a two-layer fluid model for example, it reduces to the Korteweg–de Vries (KdV) equation when there exists only one single shallow layer and to the Benjamin–Ono (BO) equation when one layer is infinitely deep and the other is shallow (Benjamin 1967; Davis & Acrivos 1967; Ono 1975).

Stationary analytic solutions of both the KdV and BO equations are well known. Korteweg & de Vries (1895) derived the ‘sech’ solitary wave and the cnoidal-wave theories for the KdV equation. Benjamin (1967) was the first to obtain the algebraic (Lorentzian) soliton as well as the periodic solution for the BO equation. An analytic expression for the soliton solution of the ILW equation was first proposed by Joseph

(1977). A discussion of Joseph's solitons and its relationship to the KdV and BO solitons is given in §2 of this paper. The existence of a periodic solution for the ILW equation was verified numerically by Kubota *et al.* (1978). Several attempts to derive analytic expressions for the ILW periodic solution, by formally introducing a purely imaginary wavenumber into the ILW soliton, have been reported (i.e. Joseph & Egri 1978; Chen & Lee 1979; Nakamura & Matsuno 1980). However, as demonstrated by Ablowitz *et al.* (1982), these analytic solutions are incorrect and are subject to certain limitations obscured by the formal methods used in the derivation. If one considers a natural periodic analogue of the ILW equation, by seeking a spatially periodic solution defined over a $(-\pi, \pi)$ instead of $(-\infty, \infty)$, one gets the PILW (periodic intermediate long-wave) equation. The PILW equation (Ablowitz *et al.* 1982; Lipovskiy 1986) was found to admit complex-valued periodic solutions which cannot be extended to the ILW equation by increasing indefinitely the wave periodic. Thus, to the best of the author's knowledge no analytic expression for the periodic solution of the ILW equations has been reported. Nevertheless, $N$-soliton solutions of the ILW equations have been obtained (Joseph & Egri 1978; Chen & Lee 1979; Matsuno 1980) and other important properties, such as infinite number of conservation laws, Bäcklund transformations, Lax pairs and inverse scattering transform schemes for numerically solving the ILW equation, are also available (see Santini *et al.* 1984). A method of deriving a periodic solution for the ILW equation is presented in §3 and the new solution thus derived is shown to reduce both to the cnoidal-wave and to the BO periodic solution under the proper limits.

In addition to the new periodic solution, another contribution of this paper is the discovery that this periodic solution of the ILW equation may be represented as an infinite sum of equally spaced identical solitons. Such a remarkable property is known to exist for the KdV equation and its relevance to the 'clean-interaction' and the 'non-destructive' characteristics of colliding KdV solitons under nonlinear coupling has been also discussed. The representation of cnoidal waves as a sum of repeated '$\text{sech}^2$' solitons, was first obtained by Toda (1975) as a by-product of a more general discussion of the 'Toda-lattice' by using an infinite product method. This interesting property of the KdV equation has also been noted and further extended by Whitham (1984). A similar property was recently found to exist also for the BO equation (Miloh & Tulin 1989) and it is demonstrated in §3 that both the KdV and BO linear superposition representations may be considered as degenerate cases of the more general ILW periodic solution.

## 2. Solitary waves

We consider the following dimensional form of the ILW equation (Joseph 1977; Kubota, Ko & Dobbs 1978)

$$U_t(x,t)+c_0\,U_x(x,t)+CU(x,t)\,U_x(x,t)+\frac{c_0\,d}{2D}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}\frac{U(x',t)\,\text{sgn}\,(x'-x)}{\exp\,(\pi|x'-x|/D)-1}\mathrm{d}x' = 0, \quad (1)$$

which is a weakly nonlinear singular integro-differential equation governing the evolution of a one-dimensional wave system. Here $c_0$ denotes the linear long-wave phase speed, $C$ is a prescribed constant representing nonlinearity and $(d, D)$ are two characteristic lengthscales, representing, for example, the corresponding depths in a two-layer model. The coefficients $c_0$, $C$ and $d$ depend in general on the density profile in the pynocline and on the eigenfunctions of a corresponding Sturm–Liouville problem (Benney 1966; Benjamin 1967; Ono 1975; Kubota *et al.* 1978). However, for a two-layer system where a shallow layer of depth $h$ and uniform density $\rho_1$ lies over an infinitely deep layer of density $\rho_2 > \rho_1$, these coefficients are given by, $c_0^2 = gh(\rho_2/\rho_1 - 1)$, $C = 3c_0/2h$ and $d = h(\rho_2/\rho_1)$ (Benjamin 1967).

The ILW equation (1) may also be considered as a particular case of the weakly nonlinear Whitham (1974) equation

$$U_t(x,t) + CU(x,t)\,U_x(x,t) + \frac{\partial}{\partial x}\int_{-\infty}^{\infty} U(\xi,t)\,G(\xi - x)\,\mathrm{d}\xi = 0, \tag{2}$$

$$G(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} c(k)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k,$$

where $c(k)$ represents the infinitesimal wave phase speed and $k$ is the wavenumber. Note that the integrals in (1) and (2) should be interpreted as Cauchy principal values when needed.

In fact it can be shown (Joseph 1977) that Whitham's equation (2) yields the ILW (1) when the phase speed is given by

$$c(k) = c_0\left[1 - \tfrac{1}{2}kd\left(\coth kD - \frac{1}{kD}\right)\right] + O(kd)^2. \tag{3}$$

This case corresponds to the so-called *thin thermocline* model (Phillips 1966) in a stable stratified fluid of total depth $D$ with a sharp thermocline centred at a depth $d$ such that $D \gg d$. If $D \rightarrow 0$ then (3) yields $c(k) - c_0 \sim k^2$ and the ILW equation (1) reduces to the KdV equation. Similarly, for $D \rightarrow \infty$, one gets from (3) $c(k) - c_0 \sim |k|$ and (1) yields the BO equation where the dispersion integral is in the form of a Hilbert transform of $U_x$. It should also be noted that the KdV shallow-water equation is valid for $d/D = O(1)$ and $\lambda/D \gg 1$, where $\lambda$ is a measure of the horizontal extent of the internal wave. On the other hand, the BO deep-water equation is valid for $\lambda/D \rightarrow 0$ and $\lambda/d \gg 1$. The ILW equation is valid therefore for a large range of $\lambda/D$ values, i.e. $\infty > \lambda/D > 0$. Further discussions on the various approximations of Whitham's long-internal-wave equation, including some experimental verifications, may be found in Miles (1980), Koop & Butler (1981), Segur & Hammack (1982) and Volyak & Krasnoslobodtsev (1986).

A stationary (permanent wave) solution of (1), in the form of $U(x,t) = U(\xi)$; $\xi = x - Vt$, must satisfy

$$-(V - c_0)\,U(\xi) + \tfrac{1}{2}CU^2(\xi) + \frac{c_0 d}{2D}\frac{\partial}{\partial \xi}\int_{-\infty}^{\infty}\frac{U(\xi')\,\mathrm{sgn}\,(\xi' - \xi)}{\exp(\pi|\xi' - \xi|/D) - 1}\,\mathrm{d}\xi' = (\tfrac{1}{2}a^2C)\,B, \tag{4}$$

where $B$ is a constant of integration, $V$ is a constant velocity of the frame of reference and $a$ is the wave amplitude defined in (5).

A solitary wave solution may be considered as a localized disturbance which decays monotonically at large distances. Hence, since $U \rightarrow 0$ as $|\xi| \rightarrow \infty$, one gets $B = 0$ in (4). A solitary solution of the homogeneous version of (4) was found by Joseph (1977) as

$$U_s(\xi) = a/[\cosh(\gamma\xi/D) + \cos\gamma]; \quad a = (d/D)\,(c_0/C)\,\gamma\sin\gamma, \tag{5}$$

where the reference velocity $V(\gamma)$ is given by

$$\frac{V}{c_0} = 1 + \frac{d}{2D}(1 - \gamma\cot\gamma), \tag{6}$$

and $\gamma$ is an arbitrary real parameter $0 < \gamma < \pi$.

The ILW soliton (5) may be also written as

$$U_s(\xi) = -\tfrac{1}{2}\mathrm{i}\gamma(d/D)(c_0/C)\left\{\tanh\tfrac{1}{2}\gamma\left(\frac{\xi}{D}+\mathrm{i}\right)-\tanh\tfrac{1}{2}\gamma\left(\frac{\xi}{D}-\mathrm{i}\right)\right\},$$

where it is interesting to note that this soliton can move either to the left or to the right.

One can define a characteristic wavelength of the solitary wave given in (5), as

$$\lambda = \frac{1}{\pi U_s(0)}\int_{-\infty}^{\infty} U_s(\xi)\,\mathrm{d}\xi = \frac{2}{\pi}D\cot(\tfrac{1}{2}\gamma), \tag{7}$$

which may be also used as a reference lengthscale instead of $D$.

The shallow depth limit of the intermediate-depth soliton (5) is found by letting in (7), $D\rightarrow 0$, and $\gamma\rightarrow 0$, such that $D/\gamma\rightarrow\frac{1}{4}\pi\lambda$, for which (5) and (6) result in the following KdV soliton (Benney 1966; Benjamin 1966);

$$U_s(\xi) = a^*\operatorname{sech}^2(\xi/\lambda^*); \quad a^* = \frac{3(V-c_0)}{C}, \tag{8}$$

with an effective wavelength $\lambda^*$ given by

$$\lambda^* = \tfrac{1}{2}\pi\lambda = \pi\left(\frac{\alpha}{V/c_0-1}\right)^{\frac{1}{2}} = \pi\left(\frac{3\alpha c_0}{aC}\right)^{\frac{1}{2}}; \quad \alpha = \tfrac{1}{6}dD. \tag{9}$$

On the other hand, for infinite depth, i.e. $D\rightarrow\infty$ and $\gamma\rightarrow\pi$, (7) yields $\sin\gamma\rightarrow\pi\lambda/D$ and the ILW soliton (5) reduces to

$$U_s(\xi) = \frac{a^*}{1+(\xi/\lambda)^2}; \quad a^* = \frac{4(V-c_0)}{C}, \tag{10}$$

which is precisely the algebraic soliton of the BO equation (Benjamin 1967) with a characteristic lengthscale given by (6) and (7), as

$$\lambda = \frac{\beta}{V/c_0-1} = \frac{4c_0\beta}{aC}; \quad \beta = \tfrac{1}{2}d. \tag{11}$$

We note that, instead of the assumption $a\lambda^2 = O(1)$ which is commonly used for formally deriving the KdV equation and its solitary wave solutions (Ursell 1953), one must instead take $a\lambda = O(1)$ when analysing the BO equation. Yet, both the KdV 'sech$^2$' soliton (8) as well as the BO algebraic soliton (10) may be considered as limiting cases for $\gamma = 0$ and $\gamma = \pi$, or $D/\lambda\downarrow 0$ and $D/\lambda\uparrow\infty$, respectively, of the generalized ILW soliton (5), which is valid for $0<\gamma<\pi$, and $\infty > D/\lambda > 0$.

## 3. Periodic waves

It is well known that both the KdV and BO equations also possess periodic solutions in addition to solitons. The periodic cnoidal wave solution of the KdV equation has been known since the turn of the century (Korteweg & de Vries 1895) and the periodic solution for the BO equation was first obtained by Benjamin (1967). In an attempt to derive a similar periodic solution for the ILW equation, Joseph & Egri (1978) postulated that such a solution may be obtained simply by letting the $\gamma$-parameter in the ILW soliton (5) be purely imaginary, which results in the following periodic solution:

$$U_p(\xi) = -\frac{(d/D)(c_0/C)\gamma\sin\gamma}{\cos(\gamma\xi/D)+\cosh\gamma}; \quad 0<\gamma<\pi.$$

Similar arguments have been employed by Chen & Lee (1979) and Nakamura & Matsuno (1980) in their quest for the ILW periodic solution. It seems that the first to note that these periodic solutions are incorrect and do not satisfy the ILW (4) equation, were Ablowitz *et al.* (1982). This may be easily verified by a direct substitution of the above solution in (4), or by examining the Fourier transform of the ILW soliton (13). It is clear that by replacing $\gamma$ by $\mathrm{i}\gamma$ the Fourier transform (13) becomes purely imaginary and no real-valued analytic solution can thus be found. In the sequel we employ a different approach and derive general periodic solutions for the ILW equation, presented here for the first time.

Towards this goal we propose to seek a periodic solution of the ILW equation (4) in the following form;

$$U_\mathrm{p}(\xi) = a\sum_{n=-\infty}^{\infty} \nu_n; \quad \nu_n = \{\cosh[2q(n-\xi/\sigma)]+\cos\gamma\}^{-1}, \quad q = \frac{\gamma\sigma}{2D}, \tag{12}$$

which is simply a linear superposition of infinite identical solitons (5), centred at $\xi = n\sigma$ for $n = 0, \pm 1, \pm 2, \ldots, \pm\infty$. Surprisingly enough, this periodic form will be shown to satisfy (4) for some particular values of $V(\gamma,\sigma)$ and $B(\gamma,\sigma)$. It is remarked here that when obtaining periodic solutions of (4) the constant of integration $B$ is generally non-zero. Furthermore, the particular choice of $B$ is not a real restriction and may also be considered as a specific normalization of the periodic solution. However, in solving (4) it is important to determine the unique value of $B$ which renders the exact solution, as in Whitham's (1984) solution of the KdV equation.

In order to show that (12) indeed satisfies (4), we first consider the Fourier transform of the ILW soliton (5) (Bateman 1954, p. 30)

$$\int_{-\infty}^{\infty} \frac{\exp(\mathrm{i}k\xi)\,\mathrm{d}\xi}{\cosh(\gamma\xi/D)+\cos\gamma} = \frac{2\pi D}{\gamma\sin\gamma}\frac{\sinh(kD)}{\sinh(\pi kD/\gamma)}. \tag{13}$$

Substituting (12) and (13) in the dispersion integral term of (4), yields (Gradshteyn & Ryzhik 1965, p. 518)

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\,k\coth(kD)\,\mathrm{e}^{-\mathrm{i}k\xi}\int_{-\infty}^{\infty}\mathrm{d}\xi' U_\mathrm{p}(\xi')\,\mathrm{e}^{\mathrm{i}k\xi'}\\
&\quad = \frac{aD}{\gamma\sin\gamma}\sum_n \int_{-\infty}^{\infty} k\frac{\cosh(kD)}{\sinh(\pi kD/\gamma)}\mathrm{e}^{\mathrm{i}k(n\sigma-\xi)}\,\mathrm{d}k\\
&\quad = \left(\frac{a}{D}\right)\gamma\cot\gamma\sum_n \nu_n + \left(\frac{a}{D}\right)\gamma\sin\gamma\sum_n \nu_n^2,
\end{aligned} \tag{14}$$

where here and elsewhere $\sum_n$ stands for $\sum_{n=-\infty}^{\infty}$.

Next, we augment (6) and express the velocity of the reference frame $V(\gamma,\sigma)$ as

$$\frac{V}{c_0} = 1+\frac{d}{2D}[1-\gamma\cot\gamma - A(\gamma,\sigma)\gamma\sin\gamma], \tag{15}$$

where $A(\gamma,\sigma)$ is a function to be determined. Comparing (12) with (5), implies that $U_\mathrm{p}\rightarrow U_\mathrm{s}$ as $\sigma\rightarrow\infty$, which shows, as expected, that $A(\gamma,\infty) = B(\gamma,\infty) = 0$. Thus, when the spacing between adjacent solitons is infinitely large, the periodic solution (12) degenerates into the ILW single soliton (5).

Finally, upon substituting (12), (14) and (15) into (4), and dividing throughout by $\frac{1}{2}(dc_0 a/D)$, the latter becomes

$$A(\gamma,\sigma)\,\gamma\sin\gamma\sum_n \nu_n + \frac{aDC}{dc_0}\Big(\sum_n \nu_n\Big)^2 - \gamma\sin\gamma\sum_n \nu_n^2 = \frac{aDC}{dc_0}B(\gamma,\sigma), \tag{16}$$

which may be further reduced, because of (5), to

$$A(\gamma,\sigma)\sum_n \nu_n + \sum\sum_{n\neq m} \nu_n\,\nu_m = B(\gamma,\sigma), \tag{17}$$

where again $\sum\sum_{n\neq m}$ denotes the double summation $\sum_n\sum_m$ excluding $m=n$.

An important and rather surprising relationship discovered for the double summation in (17) is the following (see Appendix for a proof):

$$\sum\sum_{n\neq m} \nu_n\,\nu_m = \left(-2\cos\gamma\sum_{m=1}^{\infty}[\sinh^2(qm)+\sin^2\gamma]^{-1}\right)\sum_n \nu_n + 4\sum_{m=1}^{\infty}\frac{m\coth(qm)}{\sinh^2(qm)+\sin^2\gamma}, \tag{18}$$

which holds for $\nu_n$ defined by (12).

Substituting (18) into (17) leads to

$$A(\gamma,\sigma) = 2\cos\gamma\sum_{m=1}^{\infty}\frac{1}{\sinh^2(qm)+\sin^2\gamma}, \tag{19}$$

and

$$B(\gamma,\sigma) = 4\sum_{m=1}^{\infty}\frac{m\coth(qm)}{\sinh^2(qm)+\sin^2\gamma}. \tag{20}$$

Thus, by virtue of (5), (15) and (19) it is proved that (12) is indeed a periodic solution of the ILW equation (4) (for a particular choice of $V$, $q$, $\gamma$ and $\sigma$), which may be expressed as

$$U_p(\xi) = \frac{2\gamma\sin\gamma}{K(\gamma)}\frac{(V-c_0)}{C}\sum_n\frac{1}{\cosh[2q(n-\xi/\sigma)]+\cos\gamma}, \tag{21}$$

where the parameter $K(\gamma)$ is found from (15) and (19) as

$$K(\gamma) = 1-\gamma\cot\gamma\left[1+2\sum_{m=1}^{\infty}\frac{\sin^2\gamma}{\sinh^2(qm)+\sin^2\gamma}\right]. \tag{22}$$

An alternative form of the periodic solution (21) may be found by employing the Poisson summation formula (Morse & Feshbach 1953, p. 466) which states that for any function $f(n)$, for which an exponential Fourier transform exists,

$$\sum_n f(n) = (2\pi)^{\frac{1}{2}}\sum_n F(2\pi n), \tag{23}$$

where $F(k)$ is the Fourier transform of $f(n)$. Thus, using (13) and (23) enables us to rewrite (21) in the form of the following convergent Fourier series;

$$\left.\begin{aligned} U_p(\xi) &= b\sum_n\frac{\sinh(\pi n\gamma/q)}{\sinh(\pi^2 n/q)}\,e^{2i\pi n\xi/\sigma}; \quad q=\frac{\gamma\sigma}{2D} \\ b &= \frac{2\pi\gamma}{qK(\gamma)}\frac{V-c_0}{C} = 2\pi\frac{d}{\sigma}\frac{c_0}{C}. \end{aligned}\right\} \tag{24}$$

Equation (24) constitutes an exact periodic solution of the ILW equation (4) provided that the constant of integration is given by (20). The solution depends on the environmental parameters of the thermocline, i.e. $d$, $D$, $c_0$ and $C$, on the parameter $\gamma$ which determines the reference velocity $V$ via (15) and on the spacing $\sigma$ between adjacent solitons. The mean value of $U_p(\xi)$ is therefore given by

$$\bar{U}_p = b\frac{\gamma}{\pi} = 2\gamma\frac{d\,c_0}{\sigma\,C}, \tag{25}$$

which, for supercritical wave speed $V > c_0$, represents a positive displacement of the undisturbed interface so as to increase the thickness of the thermocline. For a two-layer fluid model (which is equivalent to a delta function behaviour of the Brunt–Väisälä frequency), the mean displacement of a thermocline of depth $d$ is therefore,

$$\bar{U}_p = \tfrac{4}{3}\pi\frac{\rho_2}{\rho_1}\left(\frac{d^2}{\sigma}\right).$$

The extreme values of wave crests and wave troughs, may be determined from (24) by taking $\xi = \sigma$ and $\xi = \frac{1}{2}\sigma$ respectively. Thus, the wave amplitude is defined as

$$\hat{U}_p = 8\pi\frac{d\,c_0}{\sigma\,C}\sum_{m=1}^{\infty}\frac{\sinh[\pi(2m-1)\,\gamma/q]}{\sinh[\pi^2(2m-1)/q]}, \tag{26}$$

and the elevation midway between crests and troughs is

$$U_p^* = \bar{U}_p + 4\pi\frac{d\,c_0}{\sigma\,C}\sum_{m=1}^{\infty}\frac{\sinh(2\pi m\gamma/q)}{\sinh(2\pi^2 m/q)}. \tag{27}$$

Now, the fact that $U_p^* > \bar{U}_p$ indicates that the waves are peaked upward, i.e. they are sharper at the crests than at the troughs, as in cnoidal waves for example.

## 4. Some limiting cases

In this section we investigate a few limiting cases of (21) and (24) and compare them with some existing periodic solutions. We start with the shallow-depth limit of the ILW equation by letting $D \to 0$, $\gamma \to 0$ and $D/\gamma = \frac{1}{4}\pi\lambda$ (see equation (7)), in (21) and (22) which results in,

$$U_p(\xi) = a\sum_n \operatorname{sech}^2[(\xi - n\sigma)/\lambda^*]; \quad a = \frac{3(V-c_0)}{C}\left[1 - 6\sum_{m=1}^{\infty}\operatorname{cosech}^2(m\sigma/\lambda^*)\right]^{-1};$$
$$\lambda^* = \tfrac{1}{2}\pi\lambda. \tag{28}$$

This is an exact solution of (4) for a constant of integration $B$ given by (20),

$$B(0,\sigma) = 4\sum_{m=1}^{\infty} m\frac{\cosh(m\sigma/\lambda)}{\sinh^3(m\sigma/\lambda)}. \tag{29}$$

It is rewarding to note that (28) and (29) reduce to Whitham's (1984) periodic solution of the KdV equation, represented as a sum of canonical identical solitons, when distances are normalized with respect to the wavelength, e.g. $\lambda = 1$.

The Fourier series representation of (28) may be obtained from (24), by taking $\gamma \to 0$ and $q = \sigma/\lambda^*$ which renders

$$U_p(\xi) = (\pi\lambda^*/\sigma)^2 a\sum_n \frac{n}{\sinh(\pi^2 n\lambda^*/\sigma)}e^{2i\pi n\xi/\sigma}, \tag{30}$$

where $a$ is defined in (28). The equivalence between (28), (30) and the cnoidal-wave solution of the KdV equation has been also demonstrated by Toda (1975), Korppel & Banerjee (1981) and Boyd (1984) by using different arguments.

For the other extreme case, i.e. for deep water, one takes in (21) and (22), $D \to \infty$ and $\gamma \to \pi$, with $\sin\gamma \to \pi\lambda/D$ because of (7), which eventually leads to

$$U_p(\xi) = \frac{4(V-c_0)}{C}\left(\frac{\sigma}{2\pi\lambda}\right)\tanh\left(\frac{2\pi\lambda}{\sigma}\right)\sum_n \frac{\lambda^2}{(\xi-n\sigma)^2+\lambda^2}. \tag{31}$$

This again is identical with the periodic solution of the BO equation, expressed as a sum of repeated solitons, recently found by Miloh & Tulin (1989). It also reduces to the algebraic soliton (10) when the spacing between neighbouring solitons is infinitely large. In deriving (31), use has been made of the Mittag–Leffler expansion of 'coth $Z$' (Bromwich 1952, p. 296), i.e.

$$\coth Z = \frac{1}{Z} + \sum_{m=1}^{\infty} \frac{2Z}{(m\pi)^2+Z^2}, \tag{32}$$

which when substituted into (22) gives

$$K(\pi) \to \frac{D}{\lambda}\left(\frac{2\pi\lambda}{\sigma}\right)\coth\left(\frac{2\pi\lambda}{\sigma}\right). \tag{33}$$

For (31) to be an exact deep water solution of the ILW equation (4), that is for $D \to \infty$, the constant of integration in (4) must vanish, i.e. $B(\pi,\sigma) \to 0$, since $q \to 0$.

One may also readily find the corresponding Fourier series expansion of the BO periodic solution by letting $q \to \pi\sigma/2D$ as $D \to \infty$ in (24), which yields,

$$U_p(\xi) = \frac{2(V-c_0)}{C}\tanh\frac{2\pi\lambda}{\sigma}\sum_n \exp(-2\pi|n|\lambda/\sigma)\, e^{2i\pi n\xi/\sigma}. \tag{34}$$

Now, since
$$1+2\sum_{n=1}^{\infty}\exp(-\alpha n) = \coth(\tfrac{1}{2}\alpha); \quad \alpha > 0, \tag{35}$$

(34) may be written as the real part of

$$U_p(\xi) = \frac{2(V-c_0)}{C}\tanh\frac{2\pi\lambda}{\sigma}\coth\left[\frac{\pi}{\sigma}(\lambda - i\xi)\right] = \frac{2(V-c_0)}{C}\cdot\frac{\tanh^2\left(\frac{2\pi\lambda}{\sigma}\right)}{1-\operatorname{sech}\left(\frac{2\pi\lambda}{\sigma}\right)\cos\frac{2\pi\xi}{\sigma}}, \tag{36}$$

which is identical with Benjamin's (1967) periodic solution of the BO equation. Again, by letting $\sigma \to \infty$ in (36), the algebraic soliton (10) is recovered.

It is also noted that the ILW equation (4) yields a 'sech' type soliton which is similar in form to the envelope soliton of the nonlinear Schrödinger equation or that of the modified KdV (MKdV) equation. Such a solution corresponds to the value of $\gamma = \frac{1}{2}\pi$ and is obtained from (5) and (6) as

$$U_s(\xi) = a\operatorname{sech}\left(\frac{aC}{c_0 d}\xi\right); \quad a = \frac{\pi(V-c_0)}{C}. \tag{37}$$

In addition to the above 'sech' solution, the ILW equation also possesses for $\gamma = \frac{1}{2}\pi$ the following periodic solution (24)

$$U_p(\xi) = \frac{\pi^2}{2q}\frac{V-c_0}{C}\sum_n \operatorname{sech}\left(\frac{\pi^2 n}{2q}\right)e^{2i\pi n\xi/\sigma}; \quad q = \frac{\pi\sigma}{4D} = \frac{\pi\sigma}{2d}\left(\frac{V}{C_0}-1\right), \tag{38}$$

since $K(\frac{1}{2}\pi) = 1$, and which satisfies (4) for

$$B(\tfrac{1}{2}\pi, \sigma) = 8 \sum_{m=1}^{\infty} \frac{m}{\sinh(2qm)}. \tag{39}$$

This 'sech' solution of the ILW is believed not to have been reported before. It is also interesting to note that, in addition to the 'sech' solution, the MKdV equation also admits a Lorentzian type solution, in a similar form to (10) (Zabusky 1967, p. 223).

Finally, we discuss the case of overlapping flat solitons in the limit of $\sigma/\lambda \to 0$, or $q \to 0$. This clearly corresponds to the case of infinitesimal waves for which the periodic solution (24) yields for $\gamma < \pi$,

$$U_{\mathrm{p}}(\xi) = b\left(\frac{\gamma}{\pi} - 1\right) + b\sum_{n} \exp(-p|n|)\, \mathrm{e}^{2\mathrm{i}\pi m\xi/\sigma}; \quad p = \frac{\pi^2}{q}\left(1 - \frac{\gamma}{\pi}\right) \gg 1. \tag{40}$$

Following the same procedure that led to (36), we may show now that (40) renders

$$U_{\mathrm{p}}(\xi) - \bar{U}_{\mathrm{p}} = 2\mathrm{b}\,\mathrm{e}^{-p} \cos(2\pi\xi/\sigma), \tag{41}$$

which is the familiar infinitesimal Stokes wave solution.

## 5. Summary

A new class of periodic stationary solutions of the ILW equation (4) has been obtained. The solution is given in (24) in the form of a convergent Fourier series depending on the thermocline parameters $d$, $D$, $c_0$, $C$ and on the spacing $\sigma$ between adjacent wave crests. For infinitely large spacing, $\sigma \to \infty$, the periodic solution reduces to the ILW soliton (5) and in the other extreme case of $\sigma/\lambda \to 0$, the sinusoidal infinitesimal wave solution is recovered. The cnoidal-wave solution of the KdV, as well as Benjamin's solution of the BO equation, are also obtained as limiting cases of the general ILW solution.

It is demonstrated that the periodic solution may be represented by an infinite sum of equally spaced identical solitons. This remarkable property which was previously known to hold for both the KdV and BO equation, has now been extended and shown also to be possessed by the more general ILW equation. The analysis determines the velocity of the wavetrain and the particular value of the constant $B$ for which (21) and (24) constitute an exact solution of (4). It is remarked that the ILW soliton (5) exists only at supercritical speeds, e.g. $V > c_0$, which is implied by (6). However, (15) and the fact that $A(\gamma, \sigma) > 0$, suggests that the periodic ILW solution may appear at subcritical speeds as well.

Finally, it should be mentioned that the superposition property of solitons as a way of constructing periodic solutions of nonlinear equations, has been known to exist for other forms of evolution equations, such as the modified KdV (MKdV) and some versions of the Boussinesq equation (Whitham 1984). A similar feature has been also demonstrated by Parker (1980) for the Burgers equation, where the periodic solution is given by an infinite sum of repeated sawtooth shocks. Now, the question which remains is whether this clean-interaction property of solitons is also valid for a wider class of weakly nonlinear dispersive systems.

## Appendix. Proof of (18)

Using the definition of $\nu_n$ in (12) one can verify the relationship

$$-2\nu_n\,\nu_m = \frac{\cos\gamma}{\sinh^2(m-n)\,q+\sin^2\gamma}(\nu_n+\nu_m)+\frac{\coth(m-n)\,q}{\sinh^2(m-n)\,q+\sin^2\gamma}(w_n-w_m), \tag{A 1}$$

where
$$w_n = \nu_n \sinh\left[2q(n-\xi/\sigma)\right]. \tag{A 2}$$

Letting $n = s+r$ and $m = s-r$ in (A 1), we get

$$2\sum_{n\neq m}\sum \nu_n\,\nu_m = -\lim_{N\to\infty}\sum_{r\neq 0}\frac{\cos\gamma}{\sinh^2(rq)+\sin^2\gamma}\sum_{s=-N}^{N}(\nu_{s+r}+\nu_{s-r})$$
$$+\lim_{N\to\infty}\sum_{r\neq 0}\frac{\coth(rq)}{\sinh^2(rq)+\sin^2\gamma}\sum_{s=-N}^{N}(w_{s+r}-w_{s-r}). \tag{A 3}$$

Since $\nu_n\to 0$ as $n\to\infty$, the first double summation in the right-hand side of (A 3) is

$$-4\sum_{r=1}^{\infty}\frac{\cos\gamma}{\sinh^2(rq)+\sin^2\gamma}\sum_n \nu_n. \tag{A 4}$$

In order to evaluate the last summation in (A 3) we note that because of (A 2), $w_n\to \operatorname{sgn}(n)$ as $|n|\to\infty$, thus

$$\lim_{N\to\infty}\sum_{s=-N}^{N}(w_{s+r}-w_{s-r}) = \lim_{N\to\infty}\left(\sum_{-N+r}^{N+r} w_n-\sum_{-N-r}^{N-r} w_n\right) = 4r, \tag{A 5}$$

which completes the proof of (18).

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